Some properties of operator convex sets in spaces of linear operators

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- I. M. Gelfand, M. A. Naimark, *On the embedding of normed rings into the ring of operators in Hilbert spaces*, Math. Sbornik **12** (1943), 197-213.

C^*-algebra is a complex Banach algebra A with an involution which satisfies the following additional property

$$\|aa^*\| = \|a\|^2$$

for all $a \in A$.

- topology $\Leftrightarrow$ algebra;
- continuous map $\Leftrightarrow$ morphism;
- homeomorphism $\Leftrightarrow$ automorphism;
- measure $\Leftrightarrow$ positive functional;
- compactification $\Leftrightarrow$ unitization;
- Stonean space $\Leftrightarrow$ von Neumann algebras;
- connected $\Leftrightarrow$ projectionless.
- vector bundles $\Leftrightarrow$ projective modules;

- M. Duboise-Violette, *A generalization of the classical moment problem for $*$-algebras with applications to relativistic quantum theory I, II*, Comm. Math. Phys., **43** (1975), 225–284; ibid **54** (1977), 151–172.
- J. G. Heller, N. Neumater, S. Waldman, *A C^*-algebraic model for locally noncommutative spacetimes*, Lett. Math. Phys., **80** (2007), 257–272.
- S. Mahanta, V. Mathai, *Operator algebra quantum homogeneous spaces of universal gauge groups*, Lett. Math. Phys., **97** (2011), 263–277.
- S. Mahanta, *Twisted K-theory, K-homology, and bivariant ChernЏConnes type character of some infinite dimensional spaces*, Kyoto J. Math., **54** (2014), 3, 597–640.

- N. Steenrod, *A convenient category of topological spaces*, Mich. Math. J., **14** (1967), 133-152.

A topological vector space A, which also equipped with a ring multiplication compatible with the vector space operations such that the ring multiplication is jointly continuous is called a *topological algebra*. We say that A is a topological *-algebra if there is a continuous involution $^*: A \to A$ on A. A topological *-algebra A the topology of which is generated by a separating family $S(A)$ of submultiplicative *-seminorms is said to be *locally m-convex *-algebra* (*lmc *-algebra* for brevity). A seminorm $p \in S(A)$ is called a *C^*-seminorm* if $p(aa^*) = p(a)p(a^*)$ for all $a \in A$.

A complete lmc *-algebra is called a **pro-C^*-algebra**. We say that an element a of a C^*-algebra A is *positive* if $a = a^*$ and $\sigma(a) = [0, \infty)$, where by $\sigma(a)$ is denoted the spectrum of A.

- A. Inoue, *Locally C^*-algebras*, Mem. Faculty. Sci. Kyushu. Univ. Ser.A., **25** (1971), 197–235.
- N. C. Phillips, *Inverse limits of C^*-algebras*, J. Operator Theory, **19** (1988), 159–195.
- N. C. Phillips, *Inverse limits of C^*-algebras and applications*, Operator Algebras and Applications, Vol. 1, (ed.) D. E. Evans and M. Takesaki, London Math. Soc. Lecture Notes, No 135 (1988), 127–185.

Example

Every C^*-algebra A is a pro-C^*-algebra.

By $\mathcal{Q}$ we denote the category of all compact Hausdorff topological spaces.

Definition

Let X be a set. For a compact Hausdorff space Q by $\mathrm{Fun}(Q, X)$ we denote the set of functions from Q to X which satisfied to following conditions:

1. $\mathrm{Fun}(Q, X)$ contains all constant functions from Q to X;
2. if $f\colon Q_1 \to Q_2$ is continuous and $h\colon Q_2 \to X$ belongs to $\mathrm{Fun}(Q_2, X)$ then $h \circ f \in \mathrm{Fun}(Q_2, X)$;
3. if G is a disjoint union of $Q_1, Q_2 \in \mathcal{Q}$ then $f \in \mathrm{Fun}(G, X)$ whenever $f|Q_i \in \mathrm{Fun}(Q_i, X)$, $i \in \{1, 2\}$;
4. if $f\colon Q_1 \to Q_2$ is continuous and surjective and $h\colon Q_2 \to X$ is a function such that $h \circ f \in \mathrm{Fun}(Q_2, X)$ then $h \in \mathrm{Fun}(Q_2, X)$;

The data $(X, \{\text{Fun(Q,X)}\colon Q \in \mathcal{Q}\})$ is called a *quasitopological space* and the family $\{\text{Fun(Q,X)}\colon Q \in \mathcal{Q}\}$ is called a *quasitopology* on X.

Definition

Let X and Y be quasitopological spaces. A function $f: X \to Y$ is said to be *quasicontinuous* if for every $Q \in \mathcal{Q}$ and every $g \in \mathrm{Fun}(Q, X)$ the function $f \circ g$ belongs to $g \in \mathrm{Fun}(Q, Y)$.

We recall that a topological space Y is called *completely Hausdorff* if for any two distinct points $u, v \in Y$ there exists a continuous function $g\colon Y \to [0, 1]$ such that $g(u) = 0$ and $g(v) = 1$.

Example

Let X be a completely Hausdorff quasitopological spaces. Then the space $C(X)$ of all quasicontinuous functions on X is a pro-C^*-algebra.

The next statement characterized commutative pro-$C*$-algebras.

Theorem

The category of commutative unital σ-pro-C^*-algebras and unital homomorphisms is contravariantly equivalent to the category of countably compactly Hausdorff spaces.

- N. C. Phillips, *Inverse limits of C^*-algebras and applications*, Operator Algebras and Applications, Vol. 1, (ed.) D. E. Evans and M. Takesaki, London Math. Soc. Lecture Notes, No 135 (1988), 127–185.

Definition

Let Δ be a upward directed set, $\mathcal{H}_\alpha$ be a Hilbert space for every $\alpha \in \Delta$, $\mathcal{H}_\alpha$ be a closed subspace of $\mathcal{H}_\beta$, $\alpha \leq \beta$ and $\mathcal{H} = \varinjlim \mathcal{H}_\alpha = \bigcup_{\alpha \in \Delta} \mathcal{H}_\alpha$. Endow the vector space $\mathcal{H}$ with an inductive limit topology, that is the finest locally convex topology making the maps $i_\alpha : \mathcal{H}_\alpha \to \mathcal{H}$ continuous. Then the topological vector space $\mathcal{H}$ is called a *locally Hilbert space*.

By i_α and $i_{\alpha\beta}$ we denote the continuous embeddings $i_\alpha : \mathcal{H}_\alpha \to \mathcal{H}$ and $i_{\alpha,\beta} : \mathcal{H}_\alpha \to \mathcal{H}_\beta$, respectively. Elements of $\mathcal{H}$ can be considered as $(\xi_\alpha)_{\alpha\in\Delta}$, where $\xi_\alpha \in \mathcal{H}_\alpha$ and $\xi_\beta = i_{\alpha\beta}\xi_\alpha$ for all $\alpha, \beta \in \Delta$, $\alpha \leq \beta$. Let $\mathcal{H}$ and $\mathcal{K}$ be locally Hilbert spaces with the same index set Δ and let T be a linear operator from $\mathcal{H}$ to $\mathcal{K}$.

Since $\mathcal{H}_\alpha$ ($\mathcal{K}_\alpha$) is a closed subspace of $\mathcal{H}_\beta$ ($\mathcal{K}_\beta$) we can define the projection $P_{\alpha\beta} : \mathcal{H}_\beta \to \mathcal{H}_\alpha$ ($R_{\alpha\beta} : \mathcal{K}_\beta \to \mathcal{K}_\alpha$). Consider linear operator $T : \mathcal{H} \to \mathcal{K}$ such that $R_{\alpha\beta}T = P_{\alpha\beta}T$ for every $\alpha \leq \beta$. Put $T_\alpha := T|_{\mathcal{H}_\alpha}$. Recall that for two Hilbert spaces $\mathcal{H}_\alpha$ and $\mathcal{K}_\alpha$, the Banach space of all bounded linear operators from $\mathcal{H}_\alpha$ to $\mathcal{K}_\alpha$ is denoted by $L(\mathcal{H}_\alpha, \mathcal{K}_\alpha)$. Put

$$L(\mathcal{H}, \mathcal{K}) := \{T : \mathcal{H} \to \mathcal{K} : \ T = \varinjlim T_\alpha : T_\alpha \in L(\mathcal{H}_\alpha, \mathcal{K}_\alpha)\}$$

Let $T = \varinjlim T_\alpha \in L(\mathcal{H}, \mathcal{K})$. Consider the adjoint T_α^* of $T_\alpha \in L(\mathcal{H}_\alpha, \mathcal{K}_\alpha)$, $\alpha \in \Delta$ and let $\alpha \leq \beta$ in Δ, $x \in \mathcal{H}_\alpha$, $y \in \mathcal{H}_\alpha$. Then

$$\langle T_\beta^* x, y\rangle_\beta = \langle x, T_\beta y\rangle_\beta = \langle x, T_\alpha y\rangle_\alpha = \langle T_\alpha^* x, y\rangle_\alpha.$$

Thus we have $T_\beta^*|_{\mathcal{K}_\alpha} = T_\alpha^*$ for every $\alpha, \beta \in \Delta$, $\alpha \leq \beta$. Then there exists an unique element $T^* \in L(\mathcal{K}, \mathcal{H})$ with

$$T^* = \varinjlim T_\alpha^* \text{ such that } T^*|_{\mathcal{K}_\alpha} = T_\alpha^*, \, \alpha \in \Delta.$$

The operator T^* is called the *adjoint* of T. It is worth to note that $L(\mathcal{H}, \mathcal{K})$ has a structure of a Hilbert $L(\mathcal{H})$-module over $L(\mathcal{H})$ with the usual composition as the right module action and the inner product given by $\langle T_1, T_2 \rangle = T_1^* T_2$ for $T_1, T_2 \in L(\mathcal{H}, \mathcal{K})$. The topology on $L(\mathcal{H}, \mathcal{K})$ is generated by the family of seminorms $\| \cdot \|_\alpha$, $\alpha \in \Delta$, where

$$\| T \|_\alpha := \| T_\alpha^* T_\alpha \|_\alpha, \; \alpha \in \Delta.$$

Definition

A representation of a pro-C^*-alrebra A on a locally Hilbert space $\mathcal{H}$ is a continuous linear map $\pi : A \longrightarrow \mathcal{L}(H)$ such that

$$\pi(xy) = \pi(x)\pi(y) \text{ and } \pi(x^*) = \pi(x)^*$$

for all $x, y \in A$. A representation $\pi\colon A \longrightarrow \mathcal{L}(H)$ is said to be *nondegenerate* if $[\pi(A)(H)] = H$.

The concept of convexity plays the fundamental role in classical analysis. The noncommutative mathematics quantisation functions replaces by operators the notion of a convex set was also quantised. Loebl and Paulsen were the first who introduced the concept of an operator (C^*-convex) set.

- R. I. Loebl, V. I. Paulsen, *Some remarks on C^*-convexity*, Linear Algebra Appl., **35** (1981), 63–78.

Definition

Let A be a pro-C^*-algebra. A complex topological vector space E is said to be an A-bimodule if E is a right and a left module over A and operators: $l_a\colon E \longrightarrow E$ and $r_a\colon E \longrightarrow E$ are defined by

$$l_a(x) = ax, \;\; r_a(x) = xa, \quad x \in E$$

are continuous for every $a \in A$.

Below for simplicity we shall often use denotations ax and xa instead $l_a(x)$ and $r_a(x)$ respectively.

Example

Every pro-C^*-algebra A is evidently an A-bimodule, where l_a and r_a are left and right multiplications in A.

Definition

Let A be an unital pro-C^*-algebra and E be an A-bimodule. A subset D of E is said to be *operator convex* (or C^**-convex* in another terminology) if for every $\{x_1, \dots, x_n\} \subset E$ with $n \in \mathbb{N}$ and $\{\alpha_1, \dots, \alpha_n\} \subset A$ with $\sum_{i=1}^{m} \alpha_i^* \alpha_i = 1$ we have that $\sum_{i=1}^{m} \alpha_i^* x_i \alpha_i \in D$.

Example

Suppose A is an unital pro-C^*-algebra. Then $[0, 1_A] := \{a \in A : \ 0 \leq a \leq 1_A\}$ is an operator convex set.

Example

Let $H = \varinjlim H_\xi = \bigcup_{\xi \in \Xi} H_\xi$ be a locally Hilbert space and U be an elementary open neighbourhood of zero in $\mathcal{L}(H)$. Then U is an operator convex set.

Lemma

Suppose A is an unital pro-C^*-algebra, $\xi \in \mathbb{C}$ and $D \subset A$ is an operator convex subset of A. Then $D + \xi 1_A := \{x + \xi 1_A : x \in D\}$ is an operator convex subset as well.

Lemma

Let A be a pro-C^*-algebras, E be a A-bimodule and $D \subset E$ be an operator convex subset of E. Then $\overline{D}$ is operator convex is well.

Definition

Let A be a pro-C^*-algebra and E be an A-bimodule. By $COC(D)$ we denote the smallest closed operator convex set of E containing D.

We note that it is easy to see that $COC(D) = \bigcap_{G \in \mathfrak{D}} G$, where

$$\mathfrak{D} := \{G \subset E : \ G \text{ is closed and operator convex and } D \subset G\}.$$

Definition

Let $H = \bigcup_{\alpha \in \Xi} H_\alpha$ be a locally Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. We say that T is a *compact* operator if $T = \varinjlim T_\alpha$, where $T_\alpha \in K(H_\alpha)$ for every $\alpha \in \Xi$. The vector space of all compact operators on a locally Hilbert space H is denoted by $\mathcal{K}(H)$.

It is known that $\mathcal{K}(H)$ is a pro-C^*-algebras with respect to the family $(p_\alpha)_{\alpha \in \Xi}$ of seminorms, where

$$p_\alpha(T) = \| T|_{H_\alpha} \|_\alpha = \| T_\alpha \|_\alpha, \ \ \alpha \in \Xi.$$

Suppose $H = \bigcup_{\alpha\in\Xi} H_\alpha$. Fix $\beta \in \Xi$. By K_β we denote the orthogonally complemented space of the closed subspace $\overline{\bigcup_{\alpha<\beta}} H_\alpha$ of H_β.

Lemma

Let $H = \bigcup_{n\in\mathbb{N}} H_n$ be a locally Hilbert space such that all spaces H_n and K_n, $n \in \mathbb{N}$ are infinite dimensional and $T \in \mathcal{K}(H)$. Then $0 \in COC(T)$.

Definition

Let H be a topological vector space. A linear operator $T \in \mathcal{L}(H)$ is *not bounded below* if for every open neighbourhood of zero U in H and arbitrary open neighbourhood of zero $V \subset U$ there exists $x \in H \setminus U$ such that $Tx \in V$. We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum* if the operator $T - \lambda Id_H$ ($(T - \lambda)$ for brevity) is not bounded below on H. The approximate point spectrum of an operator $T \in \mathcal{L}(H)$ is denoted by $\sigma_a(T)$.

Lemma

Let $H = \bigcup_{n\in\mathbb{N}} H_n$ *be a locally Hilbert space such that all spaces* H_1 *and* K_n, $n \geq 2$ *are infinite dimensional and* $\lambda \in \sigma_a(T)$. *Then* $\lambda\, Id \in COC(T)$

Lemma

Let $H = \bigcup_{n \in \mathbb{N}} H_n$ *be a locally Hilbert space with* $H_n = \mathbb{C}_{k_n}$, $n \in \mathbb{N}$, $T = \varinjlim T_n \in \mathcal{L}(H)$ *and* $\lambda \in \sigma(T)$*. Then* $\lambda \, Id_H \in COC(T)$

Definition

Let A, B be pro-C^*-algebras. A linear map $\varphi\colon A \to B$ is said to be *positive*, if $\varphi(x^*x) \geq 0$ for all $a \in A$. A *completely positive map* $\varphi\colon A \to B$ of pro-C^*-algebras is a linear map such that $\varphi_n\colon M_n(A) \to M_n(B)$ defined by $\varphi_n\left((x_{ij}))_{i,j=1}^n\right) = (\varphi(x_{ij}))_{i,j=1}^n$ is positive.

We note that completely positive maps on pro-C^*-algebras were studied by different mathematicians.

The next statement gives a useful characterization of completely positive maps between pro-C^*-algebras.

Lemma

Let A and B be two pro-C^*-algebras and $\varphi\colon A \to B$ be a continuous linear map. Then φ is completely positive if and only if for any $n \in \mathbb{N}$, $\{a_1, \dots, a_n\} \subset A$ and $\{b_1, \dots, b_n\} \subset B$ we have that

$$\sum_{i,j=1}^{n} b_j^* \varphi(a_i^* a_j) b_i \geq 0 \ \text{ in } \ B.$$

A Hilbert A-module $\mathcal{M}$ over a pro-C^*-algebra A is a right A-module, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle$ that is $\mathbb{C}$-linear and A-linear in the second variable and $\langle y, x \rangle = \langle x, y \rangle^*$ $(x, y \in \mathcal{M})$ such that A is complete under the topology generated by the family of seminorms $\|x\|_\alpha = \|\langle x, x \rangle\|_\alpha^{\frac{1}{2}}$, $\alpha \in \Delta$. If the closed two-sided ideal $\langle \mathcal{M}, \mathcal{M} \rangle$ of A generated by $\{\langle x, y \rangle : x, y \in \mathcal{M}\}$ coincides with A, we say that $\mathcal{M}$ is full.

Lemma

Let A, B be pro-C^*-algebras, E be a Hilbert B-module and $\varphi\colon A \to \mathcal{L}_B(E)$ be a continuous strict completely positive linear map. Then there is a Hilbert B-module E_φ, a representation $\pi\colon A \to \mathcal{L}_B(E_\varphi)$ of A on E_φ and an element V_φ in $\mathcal{L}_B(E, E_\varphi)$ such that $\varphi(x) = V_\varphi^* \pi(x) V_\varphi$ for every $x \in A$.

- M. Joita, *Completely positive linear on pro-C^*-algebras*, University of Bucharest Press, 2008.

Definition

Let A, B be pro-C^*-algebras. A linear map $\varphi\colon A \to B$ is said to be *positive*, if $\varphi(x^*x) \geq 0$ for all $a \in A$. A *completely positive map* $\varphi\colon A \to B$ of pro-C^*-algebras is a linear map such that $\varphi_n\colon M_n(A) \to M_n(B)$ defined by $\varphi_n\left((x_{ij}))_{i,j=1}^n\right) = (\varphi(x_{ij}))_{i,j=1}^n$ is positive.

Definition

Let A, B be pro-C^*-algebras and E be a Hilbert B-module. By $\mathcal{S}_E(A)$ we denote the set of unital completely positive linear maps from A to $\mathcal{L}_B(E)$.

The next statement provides the important example of an operator convex set.

Lemma

Let A be an unital pro-C^*-algebra, B be a pro-C^*-algebra, E be a Hilbert B-module. Then $\mathcal{S}_E(A)$ is an operator convex set.

The study of completely positive linear maps is motivated by applications of the theory of completely positive linear maps to quantum information theory, where operator valued completely positive linear maps on topological algebras with an involution are used as a mathematical model for quantum operations and quantum probability.