Fourier series in a Sobolev orthogonal system of polynomials associated with Jacobi polynomials [1]

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The Sobolev-type inner product includes the derivatives of the multiplied functions. A general form of the Sobolev-type inner product can be expressed by the formula

$$\langle f, g\rangle = \sum_{k=0}^{m} \int_{\mathbb{R}} f^{(k)}(x) g^{(k)}(x) d\mu_k, \tag{1}$$

where $d\mu_k$ are Borel measures. A detailed survey of the results obtained for systems of polynomials orthogonal with respect to various Sobolev-type inner products of the form (1) can be found in [1].

[1] Marcellan, F., Xu, Y.: On Sobolev orthogonal polynomials. Expo Math. 33, 308–352 (2015)

"The lack of Christoffel-Darboux formula for Sobolev orthogonal polynomials, consequence of the lack of three-term recurrence relation, deprives an important tool for studying convergence and summability of Fourier orthogonal expansions. As a consequence, except for certain inner product of the second type and some numerical experiments, the convergence of Fourier expansions in Sobolev orthogonal polynomials has not been resolved. We consider this deficiency one of the major open problems that deserves to be studied intensively." [2]

[2] Marcellan, F., Xu, Y.: On Sobolev orthogonal polynomials. Expo Math. 33, 308–352 (2015)

Typically, the following types of Sobolev inner products are considered

- continuous (all measures μ_k, $k \geq 0$, are absolutely continuous),
- discrete (μ_0 is absolutely continuous, μ_k, $k \geq 1$, are discrete),
- discrete-continuous (μ_m is absolutely continuous, μ_k, $k < m$, are discrete).

Convergence of Fourier series in polynomials orthogonal with respect to a continuous Sobolev-type inner product are investigated in the works [1, 2, 3, 4, 5].

[1] Marcellan, F., Quintana, Y., Urieles, A.: On the Pollard decomposition method applied to some Jacobi– Sobolev expansions. Turk. J. Math. 37(6), 934–948 (2013).

[2] Ciaurri, O., Minguez, J.: Fourier series of Jacobi–Sobolev polynomials. Integral Transf. Spec. Funct. 30, 334–346 (2019).

[3] Ciaurri, O., Minguez, J.: Fourier series for coherent pairs of Jacobi measures. Preprint.

[4] B. Xh. Fejzullahu. Asymptotic properties and Fourier expansions of orthogonal polynomials with a non-discrete Gegenbauer–Sobolev inner product. Journal of Approximation Theory, Volume 162, Issue 2, 2010, Pages 397-406.

[5] B. Xh. Fejzullahu, F. Marcellan, J.J. Moreno-Balcazar, Jacobi–Sobolev orthogonal polynomials: Asymptotics and a Cohen type inequality, Journal of Approximation Theory, Volume 170, 2013, Pages 78-93.

Continuous case

In [1, 2] it is considered a case when $d\mu_k(x) = w_{\alpha+k,\beta+k}(x)dx$, where $w_{\alpha+k,\beta+k}(x) = (1-x)^{\alpha+k}(1+x)^{\beta+k}$ — Jacobi weights:

$$\langle f, g\rangle = \sum_{k=0}^{m} \int_{-1}^{1} f^{(k)}(x)g^{(k)}(x)(1-x)^{\alpha+k}(1+x)^{\beta+k}dx, \alpha, \beta > -1. \tag{2}$$

In [3] a case when $m = 1$ and measures μ_0 и μ_1 form the so-called coherent pair is investigated:

$$P_n(x;\mu_1) = \frac{P'_{n+1}(x;\mu_0)}{n+1} + a_n \frac{P'_n(x;\mu_0)}{n}, \quad n \geq 1. \tag{3}$$

In [4] necessary conditions for convergence in the norm of the Fourier series for polynomials orthogonal with respect to (1) with $m = 1$ and $d\mu_0(x) = d\mu_1(x) = (1-x^2)^{\alpha-1/2}dx$ are obtained.

In [1, 2] the questions of pointwise and uniform convergence of Fourier series in the case of a inner product of the following form are considered:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)d\mu(x) + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k) g^{(i)}(a_k), \quad (4)$$

where $d\mu(x)$ is Jacobi measure.

[1] F. Marcellan, B.P. Osilenker, I.A. Rocha, On Fourier Series of a Discrete Jacobi–Sobolev Inner Product, Journal of Approximation Theory, Volume 117, Issue 1, 2002,Pages 1-22, ISSN 0021-9045, https://doi.org/10.1006/jath.2002.3681.

[2] Rocha I. A., Marcellan F., Salto L. Relative asymptotics and Fourier series of orthogonal polynomials with a discrete Sobolev inner product // J. Approx. Theory. 2003. V. 121. P. 336–356.

Discrete case

In [1, 2, 3, 4] a special case of the above inner product is considered ($K = 2$, $N_1 = N_2 = 1$, $a_1 = -1$, $a_2 = 1$) and the convergence of Fourier series and their linear means with respect to the corresponding orthogonal polynomials are considered.

[1] B.P. Osilenker Convergence and summability of Fourier–Sobolev series // Vestnik MGSU. 2012 URL: https://cyberleninka.ru/article/n/shodimost-i-summiruemost-ryadov-furie-soboleva-1.

[2] B. P. Osilenker, On linear summability methods of fourier series in polynomials orthogonal in a discrete Sobolev space, Siberian Math. J., 2015, vol. 56, 339–351.

[3] Fejzullahu, Marcellan. On convergence and divergence of Fourier expansions with respect to some Gegenbauer-Sobolev type inner product. Communications in the Analytic Theory of Continued Fractions, 2009, n. 16, p. 1-11.

[4] Ciaurri, O., Minguez, J.: Fourier series of Gegenbauer–Sobolev polynomials. SIGMASymm. Integrabi. Geom. Methods Appl. 14, 1–11 (2018).

Fourier series in the discrete-continuous case were investigated in the works of I.I. Sharapudinov (see [1, 2] and references given there). In those works an inner product of the form

$$\langle f, g \rangle = \sum_{k=0}^{r-1} f^{(k)}(a) g^{(k)}(a) + \int_a^b f^{(r)}(x) g^{(r)}(x) \rho(x) dx. \qquad (5)$$

is considered.

[1] I.I. Sharapudinov, Sobolev-orthogonal systems of functions and some of their applications, Russian Mathematical Surveys, 2019, 74:4, 659–733.

[2] I.I. Sharapudinov, Sobolev-orthogonal systems of functions associated with an orthogonal system, Izvestiya: Mathematics, 2018, 82:1, 212–244.

Discrete-continuous case

In [1, 2, 3, 4] the problems of pointwise and uniform convergence of Fourier series in systems orthogonal with respect to (5) and associated with such classical systems as the system of Jacobi and Laguerre polynomials, the system of Haar, Walsh, Laguerre functions and the cosine system are investigated.

[1] I.I. Sharapudinov, Sobolev-orthogonal systems of functions and some of their applications, Russian Mathematical Surveys, 2019, 74:4, 659–733.

[2] I.I. Sharapudinov, Sobolev-orthogonal systems of functions associated with an orthogonal system, Izvestiya: Mathematics, 2018, 82:1, 212–244.

[3] M.G. Magomed-Kasumov, A Sobolev Orthogonal System of Functions Generated by a Walsh System, Math. Notes, 105:4 (2019), 543–549.

[4] R. M. Gadzhimirzaev, "Sobolev-orthonormal system of functions generated by the system of Laguerre functions", Issues Anal., 8(26):1 (2019), 32–46.

Discrete-continuous case

In [1] conditions are obtained that ensure the convergence in the Sobolev space of Fourier series in the system of polynomials orthogonal with respect to the discrete-continuous inner product

$$\langle f, g \rangle = \sum_{k=0}^{r-1} f^{(k)}(\omega_k) g^{(k)}(\omega_k) + \int_{-1}^{1} f^{(r)}(t) g^{(r)}(t) \rho(\alpha, \beta; t) dt,$$

where $\rho(\alpha, \beta; t) = (1-x)^{\alpha}(1+x)^{\beta}$.

[1] Diaz-Gonzalez, A., Marcellan, F., Pijeira-Cabrera, H. et al. Discrete–Continuous Jacobi–Sobolev Spaces and Fourier Series. Bull. Malays. Math. Sci. Soc. (2020).

Let us dwell in more detail on some results related to the inner product

$$\langle f, g\rangle = \sum_{k=0}^{r-1} f^{(k)}(a)g^{(k)}(a) + \int_a^b f^{(r)}(x)g^{(r)}(x)\rho(x)dx \qquad (6)$$

when $\rho(x) = \rho(\alpha, \beta; x) = (1-x)^\alpha(1+x)^\beta$.
Let $W^r_{L^p_\rho} = W^r_{L^p_\rho}[-1,1]$ be a Sobolev space, consisting of $r-1$-times continuously differentiable on $[-1,1]$ functions f such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L^p_\rho[-1,1]$, where $L^p_\rho = L^p_\rho[-1,1]$ — weighted Lebesgue space and $\rho(x) = \rho(\alpha,\beta;x)$ is Jacobi weight. For $p=2$ in the space $W^r_{L^p_\rho}$ one can introduce inner product (6).
A norm in $W^r_{L^p_\rho}$, $p \geq 1$, can be defined as follows:

$$\|f\|_{W^r_{L^p_\rho}} = \Big[\sum_{k=0}^{r-1} |f^{(k)}(a)|^p + \int_a^b |f^{(r)}(t)|^p\rho(t)dt\Big]^{1/p}. \qquad (7)$$

Let $\mathcal{P}^{\alpha,\beta} = \{\hat{P}_n^{\alpha,\beta}\}_{n=0}^{\infty}$ be a system of polynomials, orthonormed on $[-1,1]$ with Jacobi weight $\rho(\alpha,\beta;x)$ (Jacobi polynomials). Introduce new system of functions $\mathcal{P}_r^{\alpha,\beta}$, $r \geq 1$, using equalities:

$$P_{r,k}^{\alpha,\beta}(x) = \frac{(x+1)^k}{k!}, \quad k = 0, 1, \dots, r-1, \tag{8}$$

$$P_{r,k}^{\alpha,\beta}(x) = \frac{1}{(r-1)!}\int_{-1}^{x}(x-t)^{r-1}\hat{P}_{k-r}^{\alpha,\beta}(t)dt, \quad k = r, r+1, \dots. \tag{9}$$

It can be shown that such defined system will be orthonormed with respect to Sobolev inner product (6) for $\rho(x) = \rho(\alpha,\beta;x)$ [I.I Sharapudinov, Izvestiya: Mathematics, 2018].
The system $\mathcal{P}_r^{\alpha,\beta}$ is called a system of polynomials orthogonal in the Sobolev sense and associated with the Jacobi polynomials.

Fourier series of the function $f \in W^r_{L^2_{\rho(\alpha,\beta)}}[-1,1]$ in the system $\mathcal{P}_r^{\alpha,\beta}$ and the partial sum of this series can be represented as follows:

$$f(x) \sim \sum_{k=0}^{r-1} f^{(k)}(-1)\frac{(x+1)^k}{k!} + \sum_{k=r}^{\infty} c_{r,k}^{\alpha,\beta}(f) P_{r,k}^{\alpha,\beta}(x), \qquad (10)$$

$$S_{r,n}^{\alpha,\beta}(f,x) = \sum_{k=0}^{r-1} f^{(k)}(-1)\frac{(x+1)^k}{k!} + \sum_{k=r}^{n} c_{r,k}^{\alpha,\beta}(f) P_{r,k}^{\alpha,\beta}(x), \quad n \geq r, \qquad (11)$$

where $c_{r,k}^{\alpha,\beta}(f) = \int_{-1}^{1} f^{(r)}(t)\hat{P}_{k-r}^{\alpha,\beta}(t)\rho(\alpha,\beta;t)dt$.

Partial sum (11) of Sobolev-type Fourier series (10) for $n \geq r$ coincides r-times with the original function $f(x)$ at the point $x = -1$:

$$(S_{r,n}^{\alpha,\beta})^{(\nu)}(f,-1) = f^{(\nu)}(-1), \quad 0 \leq \nu \leq r-1. \qquad (12)$$

This is a very important property, which in combination with the good approximation properties of Fourier sums (11) makes them a very effective tool for the approximate solving of boundary value problems for ordinary differential equations by spectral methods. Note also that Fourier sums over classical orthogonal polynomials do not have this property.

Theorem A (Sharapudinov I.I, Izvestiya: Mathematics, 2018)

Let $\varphi_k(x)$ $(k = 0, 1, \ldots)$ form a complete in $L^2_\mu(a, b)$ orthonormed with a weight $\mu(x)$ system on a segment $[a, b]$, $\{\varphi_{r,k}(x)\}_{k=0}^\infty$ be a system, orthogonal in Sobolev sense and associated with the system $\{\varphi_k(x)\}_{k=0}^\infty$. If $\frac{1}{\mu(x)} \in L(a, b)$ and $f(x) \in W^r_{L^2_\mu(a,b)}$, then Fourier series in $\{\varphi_{r,k}(x)\}_{k=0}^\infty$ uniformly on $[a, b]$ converges to function f.

It follows that for $-1 < \alpha, \beta < 1$ the Fourier series (10) in the system, associated with Jacobi polynomials, converges uniformly on $[-1, 1]$ to the functions $f \in W^r_{L^2_{\rho(\alpha,\beta)}}[-1, 1]$ [I.I. Sharapudinov, Russian Mathematical Surveys, 2019].

A natural question arises here: whether Fourier series (10) uniformly converges for the functions $f(x) \in W^r_{L^p_{\rho(\alpha,\beta)}}[-1, 1]$ when $1 \le p < 2$.

By $U_r^{\alpha,\beta}$ we denote the set of functions f defined on $[-1,1]$ such that $\|S_{r,n}^{\alpha,\beta}(f)-f\|_C \to 0$ as $n\to\infty$.

Theorem B (Sharapudinov I.I, Izvestiya: Mathematics, 2018)

Let $A, B \in \mathbb{R}$, $p>1$ satisfy the conditions

$$\left|\frac{A+1}{p}-\frac{1}{4}\right| < \frac{1}{4}, \quad \left|\frac{B+1}{p}-\frac{1}{4}\right| < \frac{1}{4}. \tag{13}$$

Then $W^r_{L^p_{\rho(A,B)}}[-1,1] \subset U_r^{-\frac{1}{2},-\frac{1}{2}}$, $r\geq 1$.

Corollary

$W^r_{L^p_{\rho(-\frac{1}{2},-\frac{1}{2})}}[-1,1] \subset U_r^{-\frac{1}{2},-\frac{1}{2}}$, $p>1$, $r\geq 1$.

Fourier series in system $\mathcal{P}_r^{-\frac{1}{2},-\frac{1}{2}}$ can be defined for any function $f \in W^r_{L^1_{\rho(-\frac{1}{2},-\frac{1}{2})}}[-1,1]$. But above corollary is true only for functions $W^r_{L^p_{\rho(-\frac{1}{2},-\frac{1}{2})}}[-1,1]$ for $p > 1$. So question is whether the assertion of the corollary is valid for more general case when $f \in W^r_{L^1_{\rho(-\frac{1}{2},-\frac{1}{2})}}[-1,1]$. Positive answer on this question is given in the following theorem.

Theorem C (Sharapudinov I.I., Izvestiya: Mathematics, 2018)

$$W^r_{L^1_{\rho(-\frac{1}{2},-\frac{1}{2})}}[-1,1] \subset U_r^{-\frac{1}{2},-\frac{1}{2}}, \; r \geq 1.$$

An analogue of the theorem B is also proved for the system of Sobolev functions $\mathcal{P}_r^{\alpha,0} = \{P_{r,k}^{\alpha,0}\}_{k=0}^{\infty}$, associated with Jacobi polynomials $P_k^{\alpha,0}(x)$ for $-1 < \alpha \le \frac{1}{2}$, α — fractional.

Theorem D (Sharapudinov I.I., Math. Notes, 2017)

Let $-1 < \alpha \le \frac{1}{2}$, $A, B \in \mathbb{R}$, $p > 1$ *be such that*

$$\left|\frac{A+1}{p} - \frac{\alpha+1}{2}\right| < \min\left\{\frac{1}{4}, \frac{\alpha+1}{2}\right\}, \quad \left|\frac{B+1}{p} - \frac{1}{2}\right| < \frac{1}{4}.$$

Then $W^r_{L^p_{\rho(A,B)}}[-1,1] \subset U_r^{\alpha,0}$, $r \ge 1$.

The following theorem is a generalization of the theorems B, D.

Theorem 1 (M.G., Siberian Math. J., 2023)

Let $-1 < \alpha, \beta$, $A, B \in \mathbb{R}$, $p > 1$, *satisfy conditions:*

$$\left|\frac{A+1}{p} - \frac{\alpha+1}{2}\right| < \min\left\{\frac{1}{4}, \frac{\alpha+1}{2}\right\}, \tag{14}$$

$$\left|\frac{B+1}{p} - \frac{\beta+1}{2}\right| < \min\left\{\frac{1}{4}, \frac{\beta+1}{2}\right\}, \tag{15}$$

$$\frac{A+1}{p} < 1, \quad \frac{B+1}{p} < 1. \tag{16}$$

Then $W^r_{L^p_{\rho(A,B)}}[-1,1] \subset U_r^{\alpha,\beta}$, $r \geq 1$.

In fact, the previous theorem can be strengthened.

Theorem 2 (M.G., Siberian Math. J., 2023)

Let $\alpha, \beta > -1$, $A, B \in \mathbb{R}$, $p > 1$. *For any* $f \in W^r_{L^p_{\rho(A,B)}}[-1,1]$, $r \geq 1$, *Fourier series in system* $\mathcal{P}_r^{\alpha,\beta}$ *converges to* $f(x)$ *in the norm of* $W^r_{L^p_{\rho(A,B)}}[-1,1]$ *if and only if*

$$\left|\frac{A+1}{p} - \frac{\alpha+1}{2}\right| < \min\left\{\frac{1}{4}, \frac{\alpha+1}{2}\right\}, \tag{17}$$

$$\left|\frac{B+1}{p} - \frac{\beta+1}{2}\right| < \min\left\{\frac{1}{4}, \frac{\beta+1}{2}\right\}. \tag{18}$$

Under conditions (16) norm of the space $W^r_{L^p_{\rho(A,B)}}[-1,1]$ is stronger than the uniform norm, so theorem 1 follows from theorem 2.

As in the case of the system $\mathcal{P}_r^{-\frac{1}{2},-\frac{1}{2}}$, to construct the Fourier series in the system $\mathcal{P}_r^{\alpha,\beta}$ it is necessary and sufficient that $f \in W^r_{L^1_{\rho(\alpha,\beta)}}[-1,1]$. Theorem 1 is valid only for $p > 1$. The following theorem deals with the uniform convergence of Fourier series in the system $\mathcal{P}_r^{\alpha,\beta}$ for functions $f \in W^r_{L^1_{\rho(\alpha,\beta)}}[-1,1]$.

Theorem 3 (M.G., Siberian Math. J., 2023)

If $-1 < \alpha, \beta \leq 0$, *then* $W^r_{L^1_{\rho(\alpha,\beta)}} \subset U_r^{\alpha,\beta}$, $r \geq 1$.

The question of whether the conditions on α, β can be weakened remained open. At this moment we have investigated this question when $\alpha = \beta$, and for $r = 1$ necessary and sufficient conditions are obtained.

Theorem (M.G., Siberian Math. J., 2024)

Let $\alpha = \beta > -1$. Inclusion $W^r_{L^1_{\rho(\alpha,\beta)}} \subset U^{\alpha,\beta}_r$, $r \geq 1$, is valid if and only if $\alpha \leq \frac{3}{2}$.

For arbitrary $\alpha, \beta > -1$ it seems that the following theorem is true.

Theorem (Not proved yet)

Let $\alpha, \beta > -1$. *Inclusion* $W^r_{L^1_{\rho(\alpha,\beta)}} \subset U^{\alpha,\beta}_r$, $r \geq 1$, *is valid if and only if* $\alpha, \beta \leq \frac{3}{2}$ *and* $|\alpha - \beta| \leq 1$.

Thank you for the attention!