Around Strassen Desintegration Theorem

A. G. Kusraev and S. S. Kutateladze

Vladikavkaz Scientific Center of the Russian Academy of Sciences
Sobolev Mathematical Institute of the Russian Academy of Sciences

Workshop on functional analysis, approximation theory, and theory of extremal problems, dedicated to the anniversary of

Professor Vladimir Mikhaĭlovich Tikhomirov

December 5 – 6, 2024, Moscow — Novosibirsk — Vladikavkaz

Motivation

- In elementary calculus, various sufficient conditions are known for which the differentiation under the integral sign is legal:

$$\frac{d}{dx}\int_{\Omega} f(x,\omega)\,d\omega = \int_{\Omega}\frac{\partial}{\partial x} f(x,\omega)\,d\omega.$$

Motivation

- In elementary calculus, various sufficient conditions are known for which the differentiation under the integral sign is legal:

$$\frac{d}{dx}\int_{\Omega} f(x,\omega)\,d\omega = \int_{\Omega} \frac{\partial}{\partial x} f(x,\omega)\,d\omega.$$

- If $\bar{x}$ is a local extremum for the simplest variational problem

$$x(t_0) = x_0, \;\; x(t_1) = x_1, \;\; J(x(\cdot)) = \int_a^b L(t, x(t), \dot{x}(t))\,dt \to \inf,$$

then $\frac{d}{dt}J(\bar{x} + tx(\cdot))(0) = 0$ for all $x(\cdot)$ by Fermat's theorem, whence one can deduce the Euler equation:

$$-\frac{d}{dt}\bar{L}_x(t) + \bar{L}_{\dot{x}}(t), \;\; \bar{L}(t) = L(t, \bar{x}(t), \dot{\bar{x}}(t)).$$

- If $f : X \to \mathbb{R}$ is a convex function and $\bar{x} \in X$, then

$$f(\bar{x}) = \inf\{f(x) : \ x \in X\} \iff 0 \in \partial f(\bar{x}).$$

[1] A. D. Ioffe and V. M. Tikhomirov, Theory of extremal problems, Moscow, Nauka, 1974.

[2] I. Ekeland and R. Temam, Analyse convexe et problémes variationelles, Hermann, Paris, 1974.

[3] Ch. Castaing, M. Valadier, Convex analysis and measurable multifunctions Springer, Berlin etc., 1977.

- If $f : X \to \mathbb{R}$ is a convex function and $\bar{x} \in X$, then

$$f(\bar{x}) = \inf\{f(x) : \ x \in X\} \iff 0 \in \partial f(\bar{x}).$$

- To obtain a counterpart of the necessary optimality conditions in the form of the Euler equation for a convex but nondifferentiable integral functional J, some tools for calculating of subdifferential $J(\bar{x})$ are needed.

[1] A. D. Ioffe and V. M. Tikhomirov, Theory of extremal problems, Moscow, Nauka, 1974.

[2] I. Ekeland and R. Temam, Analyse convexe et problémes variationelles, Hermann, Paris, 1974.

[3] Ch. Castaing, M. Valadier, Convex analysis and measurable multifunctions Springer, Berlin etc., 1977.

- If $f : X \to \mathbb{R}$ is a convex function and $\bar{x} \in X$, then

$$f(\bar{x}) = \inf\{f(x) : \ x \in X\} \iff 0 \in \partial f(\bar{x}).$$

- To obtain a counterpart of the necessary optimality conditions in the form of the Euler equation for a convex but nondifferentiable integral functional J, some tools for calculating of subdifferential $J(\bar{x})$ are needed.
- Such tools are provided by convex analysis, a special branch of analysis that emerged in the second half of the last century and has important applications in the calculus of variations and optimal control theory, see the books[1,2,3]

[1] A. D. Ioffe and V. M. Tikhomirov, Theory of extremal problems, Moscow, Nauka, 1974.

[2] I. Ekeland and R. Temam, Analyse convexe et problémes variationelles, Hermann, Paris, 1974.

[3] Ch. Castaing, M. Valadier, Convex analysis and measurable multifunctions Springer, Berlin etc., 1977.

Convex Normal Integrands

- Convex analysis in spaces of measurable vector functions arose in connection with the needs of the theory of extremal problems and became an independent section of the convex analysis. The study of convex integral functionals was started by Rockafellar[4]; the main results are presented in the book[5].

[4] R. T. Rockafellar, Integrals which are convex functionals, Pacific J. Math. **24** (1968), 525-539.

[5] V. L. Levin, Convex analysis in spaces of measurable functions and its applications. Moscow: Nauka, 1985.

Convex Normal Integrands

- Convex analysis in spaces of measurable vector functions arose in connection with the needs of the theory of extremal problems and became an independent section of the convex analysis. The study of convex integral functionals was started by Rockafellar[4]; the main results are presented in the book[5].
- Let (Ω, Σ, μ) be a measure space and let X be a Banach space.

[4] R. T. Rockafellar, Integrals which are convex functionals, Pacific J. Math. **24** (1968), 525-539.

[5] V. L. Levin, Convex analysis in spaces of measurable functions and its applications. Moscow: Nauka, 1985.

Convex Normal Integrands

- Convex analysis in spaces of measurable vector functions arose in connection with the needs of the theory of extremal problems and became an independent section of the convex analysis. The study of convex integral functionals was started by Rockafellar[4]; the main results are presented in the book[5].
- Let (Ω, Σ, μ) be a measure space and let X be a Banach space.
- A function $f : \Omega \times X \to \mathbb{R} \cup \{+\infty\}$ is a normal convex integrand if $f_\omega(\cdot) := f(\omega, \cdot)$ is convex for each $\omega \in \Omega$ and $\omega \mapsto \operatorname{epi}(f(\omega, \cdot))$ is measurable closed-valued multifunction.

[4] R. T. Rockafellar, Integrals which are convex functionals, Pacific J. Math. **24** (1968), 525-539.

[5] V. L. Levin, Convex analysis in spaces of measurable functions and its applications. Moscow: Nauka, 1985.

Convex Integral Functionals

- Let $L^0(\Omega, X)$ be the space of (equivalence classes of) Bochner measurable X-valued function and consider two ideal function spaces $E \subset L^0(\Omega, X)$ and $F \subset L^0(\Omega, X')$. Define $I_f : E \to \mathbb{R}$

$$I_f(u(\cdot)) := \int_\Omega f_\omega(u(\omega))\, d\mu(\omega) \quad (u \in E).$$

- Let $L^0(\Omega, X)$ be the space of (equivalence classes of) Bochner measurable X-valued function and consider two ideal function spaces $E \subset L^0(\Omega, X)$ and $F \subset L^0(\Omega, X')$. Define $I_f : E \to \mathbb{R}$

$$I_f(u(\cdot)) := \int_\Omega f_\omega(u(\omega))\, d\mu(\omega) \quad (u \in E).$$

- The integral $\int_\Omega \partial f_\omega(\bar{u}(\omega))\, d\mu(\omega) \subset X'$ of a multifunction $\omega \mapsto \partial(f_\omega)$ at $\bar{u} \in E$ is defined as the set of all $\varphi \in E'$ admitting the representation

$$\langle u, \varphi \rangle = \int_\Omega \langle u(\omega), u'(\omega) \rangle\, d\mu(\omega) \quad (u \in E)$$

for some weak* measurable function $u' : \Omega \to X'$ with

$$\langle u(\cdot), u'(\cdot)) \rangle \in L^1(\Omega)\ (\forall\, u \in E), \quad u'(\omega) \in \partial f(\omega, \cdot)(\bar{u}(\omega))\ \mu\text{-a. e.}.$$

- Theorem. If X is separable and $f : \Omega \times X \to \mathbb{R}$ is a normal convex integrand, then the representation holds[6]:

$$\partial\Big(\int_{\Omega} f_{\omega}(\cdot)\,d\mu(\omega)\Big)(u) = \int_{\Omega} \partial f_{\omega}(u(\omega))\,d\mu(\omega).$$

[6] V. L. Levin, Convex analysis in spaces of measurable functions and its applications. Moscow: Nauka, 1985.

- Theorem. If X is separable and $f : \Omega \times X \to \mathbb{R}$ is a normal convex integrand, then the representation holds[6]:

$$\partial\Big(\int_\Omega f_\omega(\cdot)\,d\mu(\omega)\Big)(u) = \int_\Omega \partial f_\omega(u(\omega))\,d\mu(\omega).$$

- Define the conjugate integrand f^* by $f^*_\omega := (f_\omega)^*$ (point-wise conjugates), and put $(I_f)^*(v) := \sup_{u\in E}\{\langle\!\langle u, v\rangle\!\rangle - I_f(u)\}$, where

$$\langle\!\langle u, v\rangle\!\rangle := \textstyle\int_\Omega \langle u(\omega), v(\omega)\rangle\,d\mu \quad (u \in E,\ v \in F).$$

[6] V. L. Levin, Convex analysis in spaces of measurable functions and its applications. Moscow: Nauka, 1985.

Convex Integral Functionals

- Theorem. If X is separable and $f : \Omega \times X \to \mathbb{R}$ is a normal convex integrand, then the representation holds[6]:

$$\partial\left(\int_\Omega f_\omega(\cdot)\,d\mu(\omega)\right)(u) = \int_\Omega \partial f_\omega(u(\omega))\,d\mu(\omega).$$

- Define the conjugate integrand f^* by $f^*_\omega := (f_\omega)^*$ (point-wise conjugates), and put $(I_f)^*(v) := \sup_{u\in E}\{\langle\langle u, v\rangle\rangle - I_f(u)\}$, where

$$\langle\langle u, v\rangle\rangle := \int_\Omega \langle u(\omega), v(\omega)\rangle\,d\mu \quad (u \in E,\ v \in F).$$

- Under similar conditions, I_f and I_{f^*} are convex functionals conjugate to each other with respect to above pairing $\langle\langle\cdot,\cdot\rangle\rangle$[6]:

$$(I_f)^*(v(\cdot)) = I_{f^*}(v(\cdot)) \quad (v \in M).$$

[6] V. L. Levin, Convex analysis in spaces of measurable functions and its applications. Moscow: Nauka, 1985.

First results: References

The first results of this type in the context of convex analysis were obtained in the papers[7,8,9] and in[10] for some particular case.

It is interesting that the chain rule for the subdifferential of a convex integral functional implicitly appeared a couple of years before these papers. It was Volker Strassen's 1965 paper.

---

[7] V. L. Levin, Some properties of support functionals, Mat. Zametki, 4:6 (1968), 685-696; Math. Notes, 4:6 (1968), 900-906.

[8] A. D. Ioffe, V. M. Tikhomirov, On minimization of integral functionals, Funktsional. Anal. i Prilozhen., 3:3 (1969), 61-70; Funct. Anal. Appl., 3:3 (1969), 218-227.

[9] M. Valadier, Sous-différentiels dŠune borne supérieure et dŠune somme continue de fonctions convexes, C. R. Acad. Sci. Paris Sér. A-B Math., 268 (1969), 39-42.

[10] E. G. Gol'shtein, Problems of best approximation by elements of a convex set and some properties of the support functionals, Dokl. Akad. Nauk SSSR, 173:5 (1967), 995-998.

Strassen's Disintegration Theorem: Entourage

- Strassen's seminal disintegration theorem was established in the following environment:
  - X Separable Banach space.
  - (Ω, Σ, μ) Measure space with a complete finite measure.

Strassen's Disintegration Theorem: Entourage

- Strassen's seminal disintegration theorem was established in the following environment:
  - X Separable Banach space.
  - (Ω, Σ, μ) Measure space with a complete finite measure.
  - $p_\omega : X \to \mathbb{R}$ Continuous sublinear functional for all $\omega \in \Omega$.
  - $\omega \mapsto p_\omega(x) := p(\omega, x)$ Integrable function for all $x \in X$.
  - $\omega \mapsto \|p_\omega\| := \sup\{|p_\omega(x)| : x \in X, \|x\| \leq 1\}$ Integrable.

Strassen's Disintegration Theorem: Entourage

- Strassen's seminal disintegration theorem was established in the following environment:
  - X Separable Banach space.
  - (Ω, Σ, μ) Measure space with a complete finite measure.
  - $p_\omega : X \to \mathbb{R}$ Continuous sublinear functional for all $\omega \in \Omega$.
  - $\omega \mapsto p_\omega(x) := p(\omega, x)$ Integrable function for all $x \in X$.
  - $\omega \mapsto \|p_\omega\| := \sup\{|p_\omega(x)| : x \in X, \|x\| \le 1\}$ Integrable.
- Lemma. A sublinear function $p : X \to \mathbb{R}$ is correctly defined by

$$p(x) := \int_\Omega p_\omega(x)\, d\mu(\omega) \quad (x \in X).$$

Strassen's Disintegration Theorem: Entourage

- Strassen's seminal disintegration theorem was established in the following environment:
  - X Separable Banach space.
  - (Ω, Σ, μ) Measure space with a complete finite measure.
  - $p_\omega : X \to \mathbb{R}$ Continuous sublinear functional for all $\omega \in \Omega$.
  - $\omega \mapsto p_\omega(x) := p(\omega, x)$ Integrable function for all $x \in X$.
  - $\omega \mapsto \|p_\omega\| := \sup\{|p_\omega(x)| : x \in X, \|x\| \leq 1\}$ Integrable.
- Lemma. A sublinear function $p : X \to \mathbb{R}$ is correctly defined by

$$p(x) := \int_\Omega p_\omega(x)\, d\mu(\omega) \quad (x \in X).$$

- Strassen's disintegration theorem gives a description of the support set $\partial(p) \subset X'$:

$$\partial(p) := \{x' \in X' : \langle x, x' \rangle \leq p(x) \text{ for all } x \in x\}.$$

Strassen's Disintegration Theorem

- Theorem (Strassen; 1965)[11]. *For every* $x' \in \partial p$ *there exists a mapping* $\Omega \ni \omega \mapsto x'(\omega) \in X'$ *such that the following hold:*

  (1) $\omega \mapsto \langle x, x'(\omega)\rangle \in L_1(\Omega, \Sigma, \mu)$ *for all* $x \in X$*;*

  (2) $x'(\omega) \in \partial p_\omega$ for all $(\omega \in \Omega)$;

  (3) the representation holds:

$$\langle x, x'\rangle = \int_\Omega \langle x, x'(\omega)\rangle \, d\mu(\omega) \quad (x \in X).$$

[11] V. Strassen. *The existence of probability measures with given marginals* Ann. Math. Stat., **36** (1965), 423–439.

Strassen's Disintegration Theorem

- Theorem (Strassen; 1965)[11]. *For every $x' \in \partial p$ there exists a mapping $\Omega \ni \omega \mapsto x'(\omega) \in X'$ such that the following hold:*
  (1) $\omega \mapsto \langle x, x'(\omega)\rangle \in L_1(\Omega, \Sigma, \mu)$ *for all* $x \in X$;
  (2) $x'(\omega) \in \partial p_\omega$ for all $(\omega \in \Omega)$;
  (3) the representation holds:

$$\langle x, x'\rangle = \int_\Omega \langle x, x'(\omega)\rangle \, d\mu(\omega) \quad (x \in X).$$

- Strassen's paper was published in a journal of mathematical statistics, and it took several years for it to come to the attention of specialists in convex analysis and extremal problems.

[11] V. Strassen. *The existence of probability measures with given marginals* Ann. Math. Stat., **36** (1965), 423–439.

Strassen's Disintegration Theorem: Subdifferential Form

- Let $L^1_{w^*}(\mu, X')$ stand for all functions $x' : \Omega \to X'$ with $\omega \mapsto \langle x, x'(\omega)\rangle$ integrable for all $x \in X$) and define the integral of a multifunction $\omega \mapsto \partial p_\omega$ as

$$\int_\Omega \partial p_\omega(\cdot)\, d\mu(\omega) :=$$

$$\left\{ \int_\Omega x'(\omega)\, d\nu(\omega) : \; x'(\cdot) \in L^1_{w^*}(\mu, X'), \; x'(\omega) \in \partial p_\omega \; (\forall\, \omega \in \Omega) \right\}$$

Strassen's theorem can now be reformulated as the 'chain rule' for composite functions.

Strassen's Disintegration Theorem: Subdifferential Form

- Let $L^1_{w^*}(\mu, X')$ stand for all functions $x' : \Omega \to X'$ with $\omega \mapsto \langle x, x'(\omega)\rangle$ integrable for all $x \in X$) and define the integral of a multifunction $\omega \mapsto \partial p_\omega$ as

$$\int_\Omega \partial p_\omega(\cdot)\, d\mu(\omega) :=$$

$$\left\{ \int_\Omega x'(\omega)\, d\nu(\omega) : \ x'(\cdot) \in L^1_{w^*}(\mu, X'),\ x'(\omega) \in \partial p_\omega \ (\forall\, \omega \in \Omega) \right\}$$

  Strassen's theorem can now be reformulated as the 'chain rule' for composite functions.

- Theorem. Under the above hypothesis the representation holds:

$$\partial \left(\int_\Omega p_\omega(\cdot)\, d\mu(\omega) \right) = \int_\Omega \partial p_\omega(\cdot)\, d\mu(\omega).$$

- Assume that $L^\infty(\Omega, \Sigma, \mu)$ admits a lifting (= a right inverse of the quotient map) and X is a separable Banach space. ρ

Strassen's Disintegration Theorem: Interpretation

- Assume that $L^\infty(\Omega, \Sigma, \mu)$ admits a lifting ($=$ a right inverse of the quotient map) and X is a separable Banach space. ρ
- Define a linear functional $I_\mu : L^1(\mu) \to \mathbb{R}$ and a sublinear operator $P : X \to L^1(\mu)$ by putting

$$I_\mu := \int_\Omega u(s) d\mu(s), \quad P(x) : \left(\omega \mapsto p_\omega(x)\right)^\sim .$$

Strassen's Disintegration Theorem: Interpretation

- Assume that $L^{\infty}(\Omega, \Sigma, \mu)$ admits a lifting (= a right inverse of the quotient map) and X is a separable Banach space. ρ
- Define a linear functional $I_{\mu} : L^1(\mu) \to \mathbb{R}$ and a sublinear operator $P : X \to L^1(\mu)$ by putting

$$I_{\mu} := \int_{\Omega} u(s) d\mu(s), \quad P(x) : \big(\omega \mapsto p_{\omega}(x)\big)^{\sim}.$$

- Then evidently the representations hold:

$$p = I_{\mu} \circ P, \quad \partial(I_{\mu} \circ P) = I_{\mu} \circ \partial P. \qquad (D_1)$$

$$T \in \partial P \iff [\rho \circ T(\cdot)](\omega) \in \partial(p_{\omega})\ \omega \in \Omega. \qquad (D_2)$$

Strassen's Disintegration Theorem: Interpretation

- Assume that $L^\infty(\Omega, \Sigma, \mu)$ admits a lifting ($=$ a right inverse of the quotient map) and X is a separable Banach space. ρ
- Define a linear functional $I_\mu : L^1(\mu) \to \mathbb{R}$ and a sublinear operator $P : X \to L^1(\mu)$ by putting

$$I_\mu := \int_\Omega u(s) d\mu(s), \quad P(x) : \big(\omega \mapsto p_\omega(x)\big)^\sim.$$

- Then evidently the representations hold:

$$p = I_\mu \circ P, \quad \partial(I_\mu \circ P) = I_\mu \circ \partial P. \qquad (D_1)$$

$$T \in \partial P \iff [\rho \circ T(\cdot)](\omega) \in \partial(p_\omega)\ \omega \in \Omega. \qquad (D_2)$$

- Problem. When do (D_1) and (D_2) hold for general sublinear operators and positive linear operators?

Subdifferentials: Definition

- Definition. An operator $P : X \to E$ is said to be sublinear if

$$P(\lambda x) = \lambda P(x) \ \text{ for all } x \in X \text{ and } 0 \le \lambda \in \mathbb{R};$$
$$P(x + y) \le P(x) + P(y) \ \text{ for all } x, y \in X.$$

[12] V. L. Levin. Subdifferentials of convex mappings and of composite functions, *Siberian Math. J.*, **13**:6 (1972), 903-909.

Subdifferentials: Definition

- Definition. An operator $P : X \to E$ is said to be sublinear if

$$P(\lambda x) = \lambda P(x) \ \text{ for all } x \in X \text{ and } 0 \le \lambda \in \mathbb{R};$$
$$P(x + y) \le P(x) + P(y) \ \text{ for all } x, y \in X.$$

- Definition. A support set or a subdifferential (at zero) ∂P of a sublinear operator P is defined as $T \in \partial P \iff T \le P$:

$$\partial P := \{T \in L(X, E) : \ (\forall x \in X) \ Tx \le P(x)\}$$

[12] V. L. Levin. Subdifferentials of convex mappings and of composite functions, *Siberian Math. J.*, **13**:6 (1972), 903-909.

- Definition. An operator $P : X \to E$ is said to be sublinear if

$$P(\lambda x) = \lambda P(x) \ \text{ for all } x \in X \text{ and } 0 \leq \lambda \in \mathbb{R};$$
$$P(x + y) \leq P(x) + P(y) \ \text{ for all } x, y \in X.$$

- Definition. A support set or a subdifferential (at zero) ∂P of a sublinear operator P is defined as $T \in \partial P \iff T \leq P$:

$$\partial P := \{T \in L(X, E) : \ (\forall x \in X) \ Tx \leq P(x)\}$$

- Theorem (Levin)[12]. If $T \in L(Y, X)$ and $P : X \to F$ is sublinear then

$$\partial(P \circ T) = \partial(P) \circ T.$$

[12] V. L. Levin. Subdifferentials of convex mappings and of composite functions, *Siberian Math. J.*, **13**:6 (1972), 903-909.

Abstract Desintegration: Examples

- Theorem (Levin; 1972)[13]. If $\Phi \in L^{+}(F, \mathbb{R})$ is order continuous and $P : X \to F$ is sublinear then the following holds:

$$\partial(\Phi \circ P) = \Phi \circ \partial(P).$$

[13] V. L. Levin. Subdifferentials of convex mappings and of composite functions, *Siberian Math. J.*, **13**:6 (1972), 903-909.

[14] S. S. Kutateladze, Subdifferentials of convex operators *Siberian Math. J.*, **18**:5 (1977), 747-752.

[15] M. Neumann. On the Strassen disintegration theorem, *Arch. Math.*, **29**:4 (1977), 413-420.

Abstract Desintegration: Examples

- Theorem (Levin; 1972)[13]. If $\Phi \in L^+(F, \mathbb{R})$ is order continuous and $P : X \to F$ is sublinear then the following holds:

$$\partial(\Phi \circ P) = \Phi \circ \partial(P).$$

- Theorem (Kutatelaze; 1977)[14]. If $M := M_\phi \in \mathrm{Orth}^+(F)$ is a multiplication operator and $P : X \to F$ is sublinear then:

$$\partial(M \circ P) = M \circ \partial(P).$$

[13] V. L. Levin. Subdifferentials of convex mappings and of composite functions, *Siberian Math. J.*, **13**:6 (1972), 903-909.

[14] S. S. Kutateladze, Subdifferentials of convex operators *Siberian Math. J.*, **18**:5 (1977), 747-752.

[15] M. Neumann. On the Strassen disintegration theorem, *Arch. Math.*, **29**:4 (1977), 413-420.

Abstract Desintegration: Examples

- Theorem (Levin; 1972)[13]. If $\Phi \in L^+(F, \mathbb{R})$ is order continuous and $P : X \to F$ is sublinear then the following holds:

$$\partial(\Phi \circ P) = \Phi \circ \partial(P).$$

- Theorem (Kutatelaze; 1977)[14]. If $M := M_\phi \in \mathrm{Orth}^+(F)$ is a multiplication operator and $P : X \to F$ is sublinear then:

$$\partial(M \circ P) = M \circ \partial(P).$$

- Theorem (Neumann; 1977)[15]. Let $P_\alpha : X \to F$ be sublinear and $P(x) := \big(P_\alpha(x)\big)_{\alpha \in \mathrm{A}}$. If $\Sigma((f_\alpha)_{\alpha\in\mathrm{A}}) = \sum_{\alpha\in\mathrm{A}} f_\alpha$, then

$$\partial(\Sigma \circ P) = \Sigma \circ \partial(P).$$

---

[13] V. L. Levin. Subdifferentials of convex mappings and of composite functions, *Siberian Math. J.*, **13**:6 (1972), 903-909.

[14] S. S. Kutateladze, Subdifferentials of convex operators *Siberian Math. J.*, **18**:5 (1977), 747-752.

[15] M. Neumann. On the Strassen disintegration theorem, *Arch. Math.*, **29**:4 (1977), 413-420.

- Definition. An operator $T : E \to F$ is said to be:
  - ✓ interval preserving if $T([0, x]) = [0, Tx]$ for all $x \in E_+$;
  - ✓ order continuous if $\inf_\alpha Tx_\alpha = 0$ in F for any $x_\alpha \downarrow 0$ in E;
  - ✓ Maharam if T is order continuous and interval preserving.

[16] D. Maharam, On positive operators, Contemp. Math., **26**, 263-277 (1984).
[17] W. A. J. Luxemburg and A. R Schep, A Radon–Nikodým type theorem for positive operators and a dual, Indag. Math. (N.S.), **40**, 357-375 (1978).

Maharam operators

- Definition. An operator $T : E \to F$ is said to be:
  ✓ interval preserving if $T([0, x]) = [0, Tx]$ for all $x \in E_+$;
  ✓ order continuous if $\inf_\alpha Tx_\alpha = 0$ in F for any $x_\alpha \downarrow 0$ in E;
  ✓ Maharam if T is order continuous and interval preserving.
- In a series of papers (in 1950th), D. Maharam developed an original approach to the study of positive operators in the spaces of measurable functions. A brief description for the method and main results are collected in her survey[16].

---

[16] D. Maharam, On positive operators, Contemp. Math., **26**, 263-277 (1984).
[17] W. A. J. Luxemburg and A. R Schep, A Radon–Nikodým type theorem for positive operators and a dual, Indag. Math. (N.S.), **40**, 357-375 (1978).

- Definition. An operator $T : E \to F$ is said to be:
  - ✓ interval preserving if $T([0,x]) = [0, Tx]$ for all $x \in E_+$;
  - ✓ order continuous if $\inf_\alpha Tx_\alpha = 0$ in F for any $x_\alpha \downarrow 0$ in E;
  - ✓ Maharam if T is order continuous and interval preserving.
- In a series of papers (in 1950th), D. Maharam developed an original approach to the study of positive operators in the spaces of measurable functions. A brief description for the method and main results are collected in her survey[16].
- W. A. J. Luxemburg and A. R. Schep[17] extended a portion of Maharam's theory related to the RadonЏNikodẙym Theorem to the case of positive operators in vector lattices.

---

[16] D. Maharam, On positive operators, Contemp. Math., **26**, 263-277 (1984).
[17] W. A. J. Luxemburg and A. R Schep, A Radon–Nikodým type theorem for positive operators and a dual, Indag. Math. (N.S.), **40**, 357-375 (1978).

Abstract Desintegration: General

- Theorem (Kusraev; 1982)[18] *Let* $T : F \to G$ *be a Maharam operator and let* $P : X \to F$ *be sublinear. Then*

$$\partial(T \circ P) = T \circ \partial(P).$$

[18] A. G. Kusraev. General disintegration formulas, Dokl. Akad. Nauk SSSR, **265**:6 (1982), 1312-1316.

[19] P. Meyer–Nieberg, Strassen disintegration theorems, Arch. Math., **65** (1995), 310-315.

Abstract Desintegration: General

- Theorem (Kusraev; 1982)[18] *Let $T : F \to G$ be a Maharam operator and let $P : X \to F$ be sublinear. Then*

$$\partial(T \circ P) = T \circ \partial(P).$$

- Corollary (Meyer–Nieberg; 1995)[19]. Assume that X is a real vector space and E and F are order complete vector lattice and $E_n^{\sim}$ is point separating. Let $J \subset F^{\sim}$ be a point separating ideal and let $T : E \to F$ be a positive linear operator satisfying $T^{\sim}(J) \subset E_n^{\sim}$. If $T^{\sim}|_J$ is a lattice homomorphism, then

$$\partial(T \circ P) = T \circ \partial(P).$$

[18] A. G. Kusraev. General disintegration formulas, Dokl. Akad. Nauk SSSR, **265**:6 (1982), 1312-1316.

[19] P. Meyer–Nieberg, Strassen disintegration theorems, Arch. Math., **65** (1995), 310-315.

- A Banach bundle over Ω is a mapping $\mathcal{X} : \omega \mapsto \mathcal{X}_\omega := (\mathcal{X}_\omega, \|\cdot\|_\omega)$ with $\mathcal{X}_\omega$ a Banach space for all $\omega \in \Omega$. A section of $\mathcal{X}$ is a mapping $s : \omega \mapsto s(\omega)$ with $s(\omega) \in \mathcal{X}_\omega$ for all $\omega \in \Omega$.
- Measurable sections of $\mathcal{X}$ are defined as the limits μ-a.e. of a sequence of elements of some collection of sections given axiomatically and called a measurability structure[20].
- If a section s is measurable then the real-valued function $\omega \mapsto \|s(\omega)\|_\omega$ is also measurable and $|s|$ stands for the corresponding coset. Given $E \subset L^0(\mu)$, denote by $E(\mathcal{X})$ the space of all cosets of measurable sections of $\mathcal{X}$ with $|s| \in E$.
- Example. $s \in L^p(\mathcal{X}) \iff \int_\Omega \|s(\omega)\|_\omega^p \, d\mu(\omega) < +\infty$.

[20] A. E. Gutman, Bunach bundles in the theory of lattice normed spaces, In: Linear Operator Compatible with Order, Sobolev Institute Predd, Novosibirsk (1995), 63-211.

- Consider a measurable Banach bundle $\mathcal{X}$, and assume that:
  (Ω, Σ, μ) is measure space with the direct sum property;
  $P_\omega : X_\omega \to E$ is a continuous sublinear operator;
  $\omega \mapsto P_\omega(u(\omega))$ is Bochner μ-integrable for all $u \in E(\mathcal{X})$.
  $\omega \mapsto \|P_\omega\| := \sup\{\|P_\omega(x)\| : \|x\|_\omega \leq 1\}$ is μ-integrable.
  Then Q is a continuous dublinear operator from $E(\mathcal{X})$ to E:

$$Q(u) = \int_\Omega P(\omega, (u(\omega)))\, d\mu(\omega) \quad (u \in E(\mathcal{X}))$$

- Consider a measurable Banach bundle $\mathcal{X}$, and assume that:
  (Ω, Σ, μ) is measure space with the direct sum property;
  $P_\omega : X_\omega \to E$ is a continuous sublinear operator;
  $\omega \mapsto P_\omega(u(\omega))$ is Bochner μ-integrable for all $u \in E(\mathcal{X})$.
  $\omega \mapsto \|P_\omega\| := \sup\{\|P_\omega(x)\| : \|x\|_\omega \leq 1\}$ is μ-integrable.
  Then Q is a continuous dublinear operator from $E(\mathcal{X})$ to E:

$$Q(u) = \int_\Omega P(\omega, (u(\omega)))\, d\mu(\omega) \quad (u \in E(\mathcal{X}))$$

- Denote by $\int_\Omega \partial(P(\omega, (\cdot)(\omega))\, d\mu(\omega)$ the set of linear operators $\bar{S} : E(\mathcal{X}) \to E$ representable as

$$\bar{S}(u) = \int_\Omega S_\omega(u(\omega))\, d\mu(\omega),$$

  where $S_\omega \in \mathcal{L}(\mathcal{X}(\omega), E)$ and $S_\omega \in \partial P_\omega$ for all $\omega \in \Omega$.

Theorem (Kusraev, Kutateladze[21]; (2024)). Let (Ω, Σ, μ) be a measure space with the direct sum property, $E \subset L^0((\Omega, \Sigma, \mu))$ an ideal Banach space with order continuous norm, and $\mathcal{X}$ a liftable Banach bundle over (Ω, Σ, μ) associated with a fixed lifting of $L^\infty(\Omega)$. If the family $(P_\omega)_{\omega\in\Omega}$ satisfy the above conditions,then

$$\partial\bigg(\int\limits_\Omega P(\omega, \cdot(\omega))\, d\mu(\omega)\bigg)\partial(P) = \int_\Omega \partial(P(\omega, \cdot(\omega))\, d\mu(\omega).$$

[21] A. G. Kusraev and S. S. Kutateladze, Around Strassen desintegration theorem, to appear.

The End

Thank you for attention